J. Fluid Mech. (1966), *vol.* 24, *part* 4, *pp.* 737–751
Printed in Great Britain

On wave interactions in a stratified fluid

By S. A. THORPE

Department of Applied Mathematics and Theoretical Physics,
Cambridge University
and National Institute of Oceanography, Godalming, Surrey

(Received 1 February 1965 and in revised form 4 October 1965)

Interactions between waves in a stably stratified fluid with a free or fixed upper surface may occur at second order when the horizontal wave-numbers, $\mathbf{k}_i$, and frequencies, σ_i, of the three interacting waves satisfy the relations

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0, \quad \sigma_1 \pm \sigma_2 \pm \sigma_3 = 0.$$

These relations may be satisfied in the case when two free surface waves interact with an internal wave, or in the case when all three waves are internal, provided that they do not all belong to the same mode. The theory is applied to situations which might be realized in the laboratory.

1. Introduction

During a seminar at the National Institute of Oceanography in 1963, Dr K. Hasselmann suggested that resonant interactions might occur between trains of internal and surface gravity waves in the ocean and it was this idea that motivated the work reported in this paper.

Ball (1964) has shown that in the case of a two-fluid model, resonance is possible for second-order interactions between surface and interfacial gravity waves. In this paper the theory is extended to wave interactions in a continuously stratified fluid with particular reference to the possibility of internal wave generation.

A number of theories of internal wave generation are well known and observations of internal waves in the ocean, although few, lend support to their general conclusions. In those parts of the ocean in which currents produce a strong vertical shear, notably in straits and in the region of the Equatorial Undercurrent, internal waves have been observed (for examples see Frassetto, Backus & Hays 1962; and Metcalf, Voorhis & Stalcup 1962). These waves are probably caused by instabilities depending on the value of the local Richardson number of the kind discussed by Taylor (1931), Drazin (1958), Miles (1961) and Howard (1961). Internal waves of tidal and shorter periods have been observed on the continental shelf (Summers & Emery 1963 mention a number of these observations; see also Lafond 1959) and mechanisms by which they might be generated through the motion of surface waves over an uneven bottom have been described by Rattray (1960), and Cox & Sandstrom (1962). There is also some experimental (Sandström 1908) and theoretical (Yanowitch 1960; Cherkesov 1962) work on internal waves generated by moving surface pressures, but a general description or observational investigation of such waves has not been made. We shall discuss here a mechanism for the generation of internal gravity waves, which is quite different from any of the mechanisms mentioned, and which, for a source of energy, depends on interacting wave trains at the free surface.

Interactions between trains of surface gravity waves in a homogeneous fluid are fairly well understood (Phillips 1960; Longuet-Higgins 1962; Longuet-Higgins & Phillips 1962; Benney 1962; Hasselmann 1962, 1963 (two papers); Bretherton 1964). The interaction occurs at third order when the wave-numbers, $\mathbf{k}_i$, and frequencies, σ_i, of four wave trains satisfy the conditions,

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0,$$

and

$$\sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0.$$

We shall show that resonant interactions between internal waves and between internal and surface waves may occur, and that the interaction occurs at second order,† therefore perhaps being more powerful in its modification of wave spectra than the surface-wave interaction in a homogeneous fluid. For interactions to occur, it is necessary that three wave trains of different modes must exist in the fluid, and the wave-numbers, $\mathbf{k}_i$, and frequencies σ_i, must satisfy the conditions

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0 \quad \text{and} \quad \sigma_1 \pm \sigma_2 \pm \sigma_3 = 0. \tag{1}$$

In the case of greatest interest, two of the wave trains belong to the surface mode, whilst the third belongs to an internal mode. In this case the internal wave may be amplified by the interaction; that is, a transfer of energy from the surface to the internal wave may occur, and an internal wave generation mechanism will exist.

The main aims of this paper are to establish that resonant interaction between internal and surface waves is possible, and to demonstrate the differences between interactions in a homogeneous and in a stratified fluid. These differences are not only a result of the great complexity of modes which exist in a stratified fluid, whilst only one, the surface mode, exists in a homogeneous fluid. In the homogeneous fluid, interaction is effected through a forcing term in the free-surface boundary condition, We shall call this forcing term a surface force, although it is due to the coupling of terms of low order. In a stratified fluid the interaction is effected both through this 'surface force' and through a forcing term in the equation of motion, a 'body force'.

We do not attempt to describe the internal wave spectrum which may result from the interaction of surface waves in the ocean. Such a description, although desirable, is far beyond the scope of this paper and would require both the analysis of the modal system in a real ocean and a complete analysis of the interactions of all the possible modes of the system. Nor do we attempt to describe the resulting internal wave spectrum or the modifications to a given surface-wave spectrum in the particular example taken to illustrate the theory, that of a fluid with an exponential density profile. Cox & Sandstrom (1962) established that

† The second-order interaction of waves in the capillary and capillary-gravity part of the surface-wave spectrum has recently been studied by McGoldrick (1965).

the modal system in a fluid of such a density bears no resemblance to the real ocean system after the first few modes, and for this reason we do not further the investigation of this particular case.

2. The equations of fluid motion

The equations of motion of a non-viscous, incompressible, stably stratified fluid of non-uniform density ρ in a Cartesian frame of reference when the Boussinesq approximation†‡ has been made, may be written as:

Euler's equations, $$\rho_0 D\mathbf{u}/Dt = -\nabla p - g\rho\nabla z; \tag{2}$$

Incompressibility $$D\rho/Dt = 0; \tag{3}$$

and Continuity, $$\nabla . \mathbf{u} = 0; \tag{4}$$

where the z-axis is taken to be vertically upwards and $\mathbf{u} = (u, v, w)$ is the fluid velocity, p the pressure, g the acceleration due to gravity and ρ_0 the mean density at level z, so that the density is given by $\rho = \rho_0(z) + \rho'$. We shall suppose that the density is continuous and has continuous first and second derivatives. The effect of the Earth's rotation is neglected. Wave-type solutions of the equations are investigated, subject to the following vertical boundary conditions:

(a) the vertical velocity $w = 0$ on a horizontal bottom plane $z = -H$, and either

(b_1) $w = 0$ on a horizontal upper plane $z = 0$, or (b_2) the pressure is constant at the free surface $z = \eta(x, y, t)$.

The linearized form of the equations can be reduced to one equation for w, the vertical component of fluid velocity

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) - \mu g \nabla_1^2 w = 0, \tag{5}$$

where $$\mu = -\frac{1}{\rho_0}\partial\rho_0/\partial z, \quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad \nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2},$$

the Laplace operator. In the case of a wave-type solution $w = W(z)\exp i(\mathbf{k}.\mathbf{x} - \sigma t)$, where $\mathbf{k} = (k_x, k_y, 0)$ is the wave-number vector, σ is the wave frequency and $\mathbf{x} = (x, y, z)$, this equation reduces to an equation for the vertical velocity amplitude W,

$$\left.\begin{array}{ll} & \dfrac{d^2W}{dz^2} - k^2 W + \mu g \dfrac{k^2}{\sigma^2} W = 0, \\ \text{with boundary conditions} & \\ (a) & W = 0 \quad \text{at} \quad z = -H \\ \text{and either} \quad (b_1) & W = 0 \quad \text{at} \quad z = 0, \\ \text{or} \quad (b_2) & \dfrac{dW}{dz} = \dfrac{gk^2}{\sigma^2} W \quad \text{at} \quad z = 0.\S \end{array}\right\} \text{Set I}$$

† This is a slightly modified form of the Boussinesq approximation, taken here to simplify the algebra. Usually the ρ_0 which appears in (2) is taken to be constant.

‡ Long (1964) has shown that care must be taken in making this approximation when applying the theory to internal waves. However if the approximation is not made at this stage (and in the examples relating to the exponential density gradient studied later, $\mu H \ll 1$) our conclusions remain unaltered.

§ By imposing these boundary conditions we shall exclude the possibility of internal waves which propagate in the z-direction and $\mathbf{k}$ has therefore only horizontal components.

The boundary condition (b_2) is found by putting $p = \text{const.}$ at $z = \eta$ and using the kinematic condition $D\eta/Dt = w$ at the free surface, together with the equations of motion.

The equations of set I are well known and their implications have been studied by a number of authors, notably Fjeldstadt (1933), Groen (1948), Groen & Heyna (1958), Yih (1960) and Yanowitch (1962). (These writers did not make the Boussinesq approximation, but their results, quoted below, continue to apply here.) Using the Sturm–Liouville theory, the solutions are found to be a set of eigenfunctions $W_{1n}(z)$ with corresponding eigenvalues k_n^2, if $(k/\sigma)^2$ is held fixed. Alternatively the solutions may be expressed as a set of eigenfunctions $W_{2n}(z)$ with corresponding eigenvalues $(k_n/\sigma_n)^2$, if k^2 is held fixed, thus defining a set of frequencies σ_n corresponding to a given wave-number k. (Since the eigenvalue appears in the boundary condition (b_2), it is not possible to apply the Sturm–Liouville theory directly in this latter case, but a similar theory applies, see Yanowitch.) The sets of eigenfunctions are complete. We shall later refer to a pair (k, σ) of wave-number and frequency as being an eigenvalue of set I.

In addition to these results a theorem proved by Groen & Heyna will be used. There exists an upper bound to the frequencies of simple harmonic small-amplitude gravity waves which have their maximum amplitudes below the free surface. The upper bound is the maximum value of the quantity $\sqrt{(g\mu)}$, the stability or Brunt–Väisälä frequency in an incompressible fluid, this being the limit as k, the wave-number, tends to infinity. Thus the frequency of internal waves is bounded.

3. The equations of resonance

Following the approach of Benney (1962) in the analysis of surface-wave resonances, we look for a solution to the equations of motion of the form

$$\mathbf{u} = \sum_j \mathbf{u}_j(z,t)\, e^{i\mathbf{k}_j\cdot\mathbf{x}} \quad \text{and} \quad \rho' = \sum_j \rho_j'(z,t)\, e^{i\mathbf{k}_j\cdot\mathbf{x}}, \tag{6}$$

where

$$\mathbf{u}_j = (u_j, v_j, w_j),$$

$$\mathbf{k}_j = (k_{jx}, k_{jy}, 0),$$

and the real parts of the terms appearing on the right-hand side of the equations are understood to be taken. It will be shown that this solution may represent a system of waves, where the velocity $\mathbf{u}_j e^{i\mathbf{k}_j\cdot\mathbf{x}}$ is the contribution to the velocity field from the jth wave and $\rho_j' e^{i\mathbf{k}_j\cdot\mathbf{x}}$ is the density change caused by the jth wave.

We suppose that this solution reduces at first order to a triad of waves and look for interactions at second order between the components of the first-order triad.

Substitution into the equation of motion and reduction of the equations is carried out in Appendix 1. This leads to an equation for w_j, the vertical velocity component of the jth wave,

$$\left.\begin{aligned}&\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial z^2}w_j - k_j^2 w_j\right) - g\mu k_j^2 w_j = \begin{cases} A_{mn}(z,t) & \text{if there exist } \mathbf{k}_m, \mathbf{k}_n, \text{ such that } \mathbf{k}_j \pm \mathbf{k}_m \pm \mathbf{k}_n = 0, \\ 0 & \text{otherwise,}\end{cases}\\ &\text{with boundary conditions}\\ &\quad (a)\ \ w_j = 0 \quad \text{at} \quad z = -H\\ &\text{and either}\\ &\quad (b_1)\ \ w_j = 0 \quad \text{at} \quad z = 0\\ &\text{or}\\ &\quad (b_2)\ \ \frac{\partial^2}{\partial t^2}\frac{\partial w_j}{\partial z} + gk_j^2 w_j = \begin{cases} B_{mn}(t) & \text{at } z = 0 \quad \text{if there exist } \mathbf{k}_m, \mathbf{k}_n \text{ such that } \mathbf{k}_j \pm \mathbf{k}_m \pm \mathbf{k}_n = 0, \\ 0 & \text{at } z = 0 \quad \text{otherwise.}\end{cases}\end{aligned}\right\}\ \text{Set II}$$

A_{mn} and B_{mn} are second-order sums of products of terms arising from the components of $\mathbf{u}_m$, $\mathbf{u}_n$, ρ'_m, ρ'_n and similar terms proportional to the surface disturbances and pressure disturbances of the mth and nth wave trains.

Now suppose that the wave-numbers $\mathbf{k}_l, \mathbf{k}_m, \mathbf{k}_n$ of the first-order triad do satisfy the relation

$$\mathbf{k}_l \pm \mathbf{k}_m \pm \mathbf{k}_n = 0. \tag{7}$$

If a wave-type solution of the problem is possible, we may write

$$\mathbf{u}_m = \mathbf{U}_m(z,t)\, e^{-i\sigma_m t} \quad \text{and} \quad \rho'_m = \Pi_m(z,t)\, e^{-i\sigma_m t}, \tag{8}$$

where (k_m, σ_m) is an eigenvalue of set I (and similarly for $\mathbf{u}_n$, etc.). (It was shown in §2 that, given k, a set of frequencies may be found so that (k, σ) is an eigenvalue. σ_m is any one of the possible set of frequencies corresponding to k_m.) Now the terms A_{mn} and B_{mn} of set II may be written in the form

$$\begin{aligned}A_{mn} &= a^+_{mn}(z,t)\, e^{-i(\sigma_m+\sigma_n)t} + a^-_{mn}(z,t)\, e^{-i(\sigma_m-\sigma_n)t}\\ &= a^\pm_{mn}(z,t)\, e^{-i(\sigma_m\pm\sigma_n)t},\end{aligned} \tag{9}$$

say, and

$$B_{mn} = b^\pm_{mn}(t)\, e^{-i(\sigma_m\pm\sigma_n)t}, \tag{10}$$

where $a^\pm_{mn}$ contain second-order products of pairs of components of $\mathbf{U}_m$, $\mathbf{U}_n$, Π_m, Π_n, etc., and $b^\pm_{mn}$ are defined in a similar way but evaluated at $z = 0$.

Hence the problem is reduced to one of solving the set of equations

$$\left.\begin{aligned}&\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_l}{\partial z^2} - k_l^2 w_l\right) - g\mu k_l^2 w_l = a^\pm_{mn}(z,t)\, e^{-i(\sigma_m\pm\sigma_n)t}, \quad \text{III (i)},\\ &\text{with boundary conditions}\\ &\qquad (a)\ \ w_l = 0 \quad \text{at} \quad z = -H\\ &\text{and either}\\ &\qquad (b_1)\ \ w_l = 0 \quad \text{at} \quad z = 0\\ &\text{or} \qquad (b_2)\ \ \frac{\partial^2}{\partial t^2}\frac{\partial w_l}{\partial z} + gk_l^2 w_l = b^\pm_{mn}(t)\, e^{-i(\sigma_m\pm\sigma_n)t} \quad \text{at} \quad z = 0,\end{aligned}\right\}\ \text{Set III}$$

for w_l of the form $W'_l(z,t)\, e^{-i\sigma_l t}$.

W'_l is the amplitude of the vertical velocity component of the lth wave as a function of time and vertical co-ordinate. The terms on the right-hand side of set III represent interactions between other waves of the system which have the same wavelength as the lth wave. We wish to examine circumstances in which these interactions can contribute towards the amplitude of the lth wave.

It will be noticed that III (i) contains an interaction term on the right-hand side, and this is equivalent to an effective 'body force' acting on the system in addition to the 'surface force' which may act at the free surface and which is represented on the right-hand side of (b_2). This body force is quite absent in the case of interacting surface waves in a homogeneous fluid, since product (or interaction) terms do not occur in the governing equation of motion, the Laplace equation. (The second-order terms representing the effective body force are partly a result of the vorticity generation in the fluid of non-constant density, as may be seen from examination of (A 2), Appendix 1.)

The solutions of set III are discussed in Appendix 2. It is shown that if there are waves in the system with wave-numbers and frequencies such that the resonance conditions

$$\mathbf{k}_l = \pm \mathbf{k}_m \pm \mathbf{k}_n, \quad \text{and} \quad \sigma_l = \pm \sigma_m \pm \sigma_n \tag{11}$$

are satisfied, then interaction may occur provided that certain coefficients are non-zero, and the amplitude of each of the three waves will slowly change with time.

If (k_m, σ_m) and (k_n, σ_n) are eigenvalues of set I corresponding to a surface wave mode and (k_l, σ_l) is an eigenvalue of an internal mode, then the interaction represents a transfer of energy between two surface waves and an internal wave; if all the eigenvalues correspond to internal modes, the interaction represents an energy transfer between internal waves.

It may be shown algebraically that in a fluid of general stable density structure it is possible to find a pair of surface waves and an internal wave which interact. It is, however, simpler to argue from a diagram (figure 1) demonstrating the interactions, similar to that used by Ball (1964) in the discussion of interactions in a two-fluid system. The 'cone' E is the locus of surface wave-numbers and frequencies $(k_x, k_y, \sigma) \equiv (\mathbf{k}, \sigma)$ say. A is a point on E representing a particular surface wave ($\mathbf{k}_m, \sigma_m$). The set of complete 'cones' I_1, I_2, I_3, ... (only three are shown) represent internal wave modes, each one representing a different mode, but with their origin translated from O to the point A, so that any point on one of these cones having an internal wave-number and frequency ($\mathbf{k}_l, \sigma_l$) will, in the translated co-ordinates shown, be at the position ($\mathbf{k}_l + \mathbf{k}_m$, $\sigma_l + \sigma_m$). Hence the points of intersection of the cones $I_1, I_2, I_3, \ldots$ with E will represent surface waves ($\mathbf{k}_n, \sigma_n$) such that $\mathbf{k}_n = \mathbf{k}_l + \mathbf{k}_m$ and $\sigma_n = \sigma_l + \sigma_m$ for some internal wave ($\mathbf{k}_l, \sigma_l$) and, these being the resonance conditions, the points of intersection of $I_1, I_2, I_3, \ldots$ and E represent surface waves which may interact with the surface wave represented by A and an internal wave. The difference between surface wave-numbers and frequencies must be taken to satisfy the resonance conditions. The planes P_1, P_2 are given by $\sigma = \sigma_m \pm \sqrt{(g\mu_{\max})}$, where $\mu_{\max}$ is the maximum value of μ, and by Groen's Theorem, the two planes are the limits between which $I_1, I_2, I_3, \ldots$ must lie. Since $\sqrt{(g\mu_{\max})}$ is usually very small in the ocean compared with surface-wave frequencies, the planes P_1 and P_2 are not far separated, and the curves of intersection of $I_1, I_2, I_3, \ldots$ and E do not differ greatly from circles. For interaction it is therefore necessary that surface waves of almost equal wave-numbers must intersect at some angle θ, and internal waves of long wavelength will take part in the interaction if θ is small.

It is also easy to demonstrate graphically that a set of three internal waves may be found which satisfy the resonance conditions, provided that the waves do not all belong to the same mode.

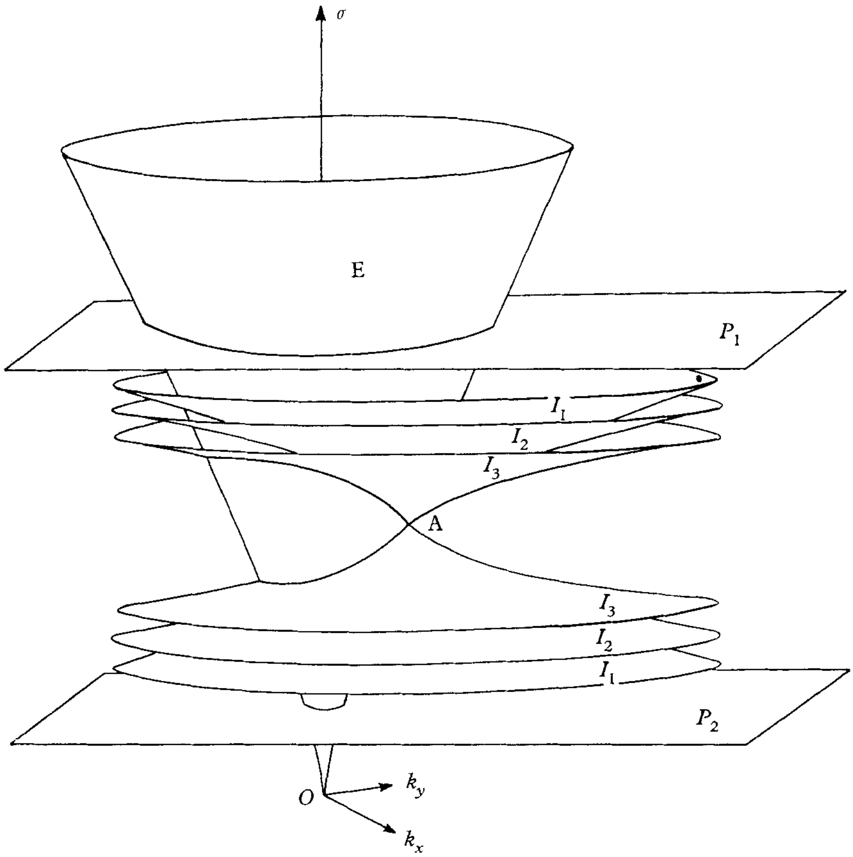

FIGURE 1. Interaction diagram. The curves of intersection of the internal mode cones $I_1, I_2, I_3, \ldots$, with the surface mode cone E, represent surface waves which can interact with the surface wave represented by the point A and an internal wave.

4. Examples

The results of the analysis carried out in Appendices 1 and 2 have been applied to two cases to obtain some idea of the importance of the interactions. It has been assumed that the density is an exponential function of depth so that μ is constant, and that the density difference between the top and bottom of the fluid is small compared with the mean density ($\mu H \ll 1$). The surface waves are supposed to be in phase (so that f_1 and f_2 in equation (A9) are real) and in deep water (so that $\tanh k_i H = 1$, $i = 1, 2$). In this case the eigenfunctions of set I of the surface mode are proportional to $\sinh k_i(z+H)$ and those of the internal modes are proportional to $\sin \lambda_n(z+H)$, where n is the number of the internal mode. λ_n is approximately equal to $n\pi/H$ (see Lamb 1932, §235). It is also assumed that the initial growth rate of a wave resulting from the interaction of two others is maintained, so that variation in the wave amplitudes of the other waves is neglected. This assumption is probably valid if the change of wave amplitude of an internal wave resulting from the interaction between it and a pair of surface waves is considered, at least until the amplitude of the internal wave is comparable to that of the surface waves, since the energy density of an internal wave is much less than that of a surface wave of equal amplitude if the density gradient is small, and thus the energy extracted from the surface waves, and therefore the change in their amplitudes, will be small.

Case 1

Suppose that a pair of surface waves of amplitudes a_1, a_2, and wave-numbers k_1, k_2, intersect at an angle θ, and interact together with an internal wave of the nth mode. Then, if $(\mu H)(k_i H)$ is not large compared with unity for $i = 1, 2$, and the stability frequency, $\surd(g\mu)$, is much less than the frequency of the surface waves (which implies that for interaction k_1 is approximately equal to k_2), it is found that the rate of change of the maximum amplitude of the internal wave motion, $\dot{A}$, is given approximately by

$$\dot{A} = \frac{(a_1 k_1)(a_2 k_2)(g\mu)^{\frac{1}{2}} Hn\pi \sin\frac{1}{2}\theta}{(n^2\pi^2 + k^2H^2)^{\frac{3}{2}}} \times \left\{3 + 2\cos\theta + (-1)^{n+1} . 3 . \frac{k^2H^2 + n^2\pi^2}{4k_1^2H^2 + n^2\pi^2}(1 + \cos\theta)\right\},$$

where k is the internal wave-number, approximately $2k_1 \sin\frac{1}{2}\theta$. (In deriving this expression it is found that the surface and body forces of the interaction play roles which are comparable in magnitude. The terms $3 + 2\cos\theta$ in the major brackets in the expression for $\dot{A}$ arise from the surface boundary condition whilst the remainder of the terms in the brackets represent the effect of the non-linearity of the equation of motion.)

For example, if $n = 1$ and $k_1 H = \pi$, the maximum amplification rate occurs for surface waves intersecting at about 45° when

$$\dot{A} = 0{\cdot}36 a_1 a_2 k_2 (g\mu)^{\frac{1}{2}}$$

and if $(g\mu)^{\frac{1}{2}} = 1\,\text{sec}^{-1}$ and $a_2 k_2$, the maximum slope of the second surface-wave train, is 0·1, then the internal wave amplitude will be equal to that of the first surface wave after an interaction time of about 28 sec.

For a laboratory experiment, it is usually not the amplification rate which is significant, but the amplitude which might be achieved by an internal wave after the continued interaction of surface waves over a fetch, D. This wave amplitude, $A(D)$ is equal to $\dot{A}D/c_g$, where c_g is the group velocity of the internal waves, and is given by

$$A(D) = \frac{a_1 a_2 k_2 k D}{2n\pi}\left\{3 + 2\cos\theta + (-1)^{n+1} . 3 . \frac{k^2H^2 + n^2\pi^2}{4k_1^2H^2 + n^2\pi^2}(1 + \cos\theta)\right\}.$$

If $k_1 H = \pi$ and $n = 1$, the maximum wave amplitude results from the interaction of surface waves intersecting at about 88°. For $\theta = 90°$, close to the maximum,

$$A(D) = 1{\cdot}08(a_1 k_1)(a_2 k_2) D,$$

which is large, for if the surface wave slopes are only 0·1, the internal wave amplitude will be over a centimetre after interaction over a fetch of only one metre. The practical difficulties of testing the prediction experimentally are, however, considerable, since the surface waves must be dissipated without significantly affecting the density gradient. If $k_1 H = 2\pi$ and $n = 1$, the maximum of $A(D)$ is found at $\theta = 95^\circ 15'$; for $\theta = 90^\circ$, $A(D) = 1{\cdot}03(a_1 k_1)(a_2 k_2) D$.

Case 2

This is the case of interaction between three parallel internal waves. To give the example physical significance, it is supposed that waves are generated in a channel by the oscillatory motion of a flap hinged about a horizontal axis at mid-depth $z = -\frac{1}{2}H$. In this case all the waves generated have the same frequency, σ, and provided that the flap motion is small, all the waves belong to the odd modes (so that n is odd), and the amplitudes of the waves may easily be found in terms of the flap amplitude.†

The following results are found.

1. The resonance conditions may be satisfied, given two waves of frequency σ, and horizontal wave-numbers k_1, k_2, and model numbers n_1, n_2, only by the presence of a third wave of frequency 2σ, wave-number $k_1 + k_2$ and model number $n_1 - n_2$. The interactions between the set of odd modes generated by the flap result in the generation of even modes. (Experimentally these would grow with distance down the channel and would be quite distinct from any even modes generated by the finite motion of the flap.)

2. For a given value of the stability frequency $\surd(g\mu)$, interaction may occur only for certain values of the frequency, σ, given by

$$\frac{g\mu}{\sigma^2} = 1 + \frac{3(2M+1+N)^2}{(2M+1-N)(2M+1+3N)},$$

where M and N are positive integers and $2M+1$, $2M+1+2N$ and $2N$ are the model numbers of the interacting triad.

3. The lowest even mode grows most rapidly as a result of the interaction. The wave of largest amplitude which may be produced as a result of interaction over a distance D (on the assumption that the initial growth rate is maintained) is that of the second mode resulting from the interaction of the 3rd and 5th modes when $g\mu/\sigma^2 = 5$. The maximum amplitude of the wave occurs at levels $z = -\frac{1}{4}H$, $-\frac{3}{4}H$ and is $\frac{16}{25}(a^2/H^2) D$.

5. Conclusion

Resonant interactions may occur at second order between a pair of surface waves and an internal wave in a stably stratified fluid, and interaction may also occur between three internal waves of different modes. The interaction coupling is made through effective body and surface forces which result from the presence of the waves when the resonance conditions are satisfied. The calculations have

† If a is the amplitude of the flap at the upper boundary of the fluid, then the maximum amplitude of the nth wave mode is $8a/\{(g\mu/\sigma^2 - 1)^{\frac{1}{2}} n^2\pi^2\}$.

been applied to determine the magnitude of waves which would result from interactions in situations which might be realized in the laboratory, but a description of interactions in a continuous spectrum of waves belonging to many different internal and surface modes in a fluid of density structure resembling that of the ocean has not been attempted. Far more theoretical, experimental, and observational work remains to be done before any conclusion may be made about the importance of wave interactions in modifying the spectra of ocean waves, but that such complex interactions exist between waves in the ocean thermocline and those at the sea surface is possible, and internal waves may well be generated by surface wave interactions.

An inaccuracy in Ball's (1964) paper on interactions between surface and interfacial waves in a two-fluid model is noted in Appendix 3.

It is a pleasure to express my thanks for the help and encouragement of Dr O. M. Phillips, who was my supervisor during the time that much of this work was done, and to Dr F. P. Bretherton, with whom I have had several enlightening discussions.

Appendix 1

Look for a solution of equations (2), (3), (4) of the form

$$\mathbf{u} = \sum_j \mathbf{u}_j(z,t)\, e^{i\mathbf{k}_j\cdot\mathbf{x}}, \quad \rho' = \sum_j \rho'_j(z,t)\, e^{i\mathbf{k}_j\cdot\mathbf{x}}, \tag{A 1}$$

where real parts are to be taken, and $\mathbf{u}_j = (u_j, v_j, w_j)$.

Combining (2), (3) and (4) and substituting for $\mathbf{u}$, ρ', we find

$$\sum_j ik_{jy}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial z^2}w_j - k_j^2 w_j\right) - \mu k_j^2 w_j\right] e^{i\mathbf{k}_j\cdot\mathbf{x}} = S_1, \tag{A 2}$$

where

$$S_1 = \frac{\partial^3}{\partial x\,\partial z\,\partial t}\{(\mathbf{w}\,.\,\nabla)\,w - (\mathbf{u}\,.\,\nabla)\,w_z\} + \nabla_1^2\left\{\frac{\partial}{\partial t}[(\mathbf{w}\,.\,\nabla)\,u - (\mathbf{u}\,.\,\nabla)\,w_x] - \frac{g}{\rho_0}\frac{\partial}{\partial y}[(\mathbf{u}\,.\,\nabla)\,\rho']\right\},$$

and

$$\mathbf{w} = \nabla\times\mathbf{u} = (w_x, w_y, w_z).$$

Each of the terms on the right-hand side of (A 2) is a second-order sum of the products of the components of the real parts of $\mathbf{u}_j e^{i\mathbf{k}_j\cdot\mathbf{x}}$, their derivatives, and the real part of $\rho'_j\, e^{i\mathbf{k}_j\cdot\mathbf{x}}$ and their derivatives. Hence the right-hand side of (A 2) may be represented by a sum

$$\sum_l \alpha_l(z,t)\, e^{i\mathbf{k}_l\cdot\mathbf{x}},$$

where l is summed over all wave-numbers $\mathbf{k}_l$ such that $\mathbf{k}_l = \pm\,\mathbf{k}_m \pm \mathbf{k}_n$ and m and n are summed over all the primary wave-numbers present. $\alpha_l(z,t)$ is second order, at least. (A 2) represents the forcing of the first-order velocity components (the terms on the left-hand side) by the second-order interactions, acting as an effective body force in the fluid.

If for some $\mathbf{k}_j$ there exist $\mathbf{k}_m$ and $\mathbf{k}_n$ such that $\mathbf{k}_j = \pm\,\mathbf{k}_m \pm \mathbf{k}_n$ then for this $\mathbf{k}_j$ we may take components of (A 2) and write

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial z^2}w_j - k_j^2 w_j\right) - g\mu k_j^2 = A_{mn}(z,t), \tag{A 3}$$

where $A_{mn} = \alpha_j/ik_{jy}$.

The boundary condition (b_2) that p = const., P say, at the free surface $z = \eta(x, t)$, is evaluated in terms of w_j. Let the pressure $p = p_0 + p'$, where

$$p_0(z) = P - \int_z^0 g\rho_0(z')\,dz'. \tag{A 4}$$

Now at the free surface $z = \eta$, the pressure is constant, so

$$p_0(\eta) + p'(x, y, \eta, t) = P,$$

and expanding about $z = 0$

$$p_0 + \eta\frac{dp_0}{dz} + \tfrac{1}{2}\eta^2\frac{d^2p_0}{dz^2} + p' + \eta\frac{\partial p'}{\partial z} + \text{higher orders} = P \quad \text{at} \quad z = 0,$$

and hence

$$\begin{aligned}\eta\frac{dp_0}{dz} + p' &= -\left[\tfrac{1}{2}\eta^2\frac{d^2p_0}{dz^2} + \eta\frac{\partial p'}{\partial z} + \text{higher orders}\right] \quad \text{at} \quad z = 0 \\ &= S_2, \quad \text{say.}\end{aligned} \tag{A 5}$$

Expanding the kinematic condition

$$\partial\eta/\partial t + (\mathbf{u}\,.\,\nabla)\,\eta = w \quad \text{at} \quad z = \eta \tag{A 6}$$

about $z = 0$ and combining this with (A 5) and using the equations of motion and (A 1) we find

$$\sum_j ik_{jx}\left[\frac{\partial^2}{\partial t^2}\frac{\partial w_j}{\partial z} + gk_j^2 w_j\right]e^{i\mathbf{k}_j\cdot\mathbf{x}} = S_3 \quad \text{at} \quad z = 0, \tag{A 7}$$

where

$$S_3 = -\frac{\partial^2}{\partial y\,\partial t}[(\mathbf{w}\,.\,\nabla)\,w - (\mathbf{w}\,.\,\nabla)\,w_z] + \nabla_1^2\left[\frac{1}{\rho_0}\frac{\partial^2 S_2}{\partial x\,\partial t} - g\frac{\partial}{\partial x}(\mathbf{u}\,.\,\nabla\eta) + \frac{\partial}{\partial t}(\mathbf{u}\,.\,\nabla u)\right].$$

Now expand η in the form $\eta = \sum_j \eta_j e^{i\mathbf{k}_j\cdot\mathbf{x}}$, where $\eta_j(t)$ is the amplitude of the surface disturbance of the jth wave, and pressure, p', into similar components due to the wave motion, $p' = \sum_j p'_j e^{i\mathbf{k}_j\cdot\mathbf{x}}$. Then if we include on the right-hand side of (A 7) only terms of second order at most, the right-hand side is a second-order sum of products of pairs of components of velocity, pressure, density and surface displacement and their derivatives, and as before, if $\mathbf{k}_j = \pm\mathbf{k}_m \pm \mathbf{k}_n$ for some $\mathbf{k}_m$, $\mathbf{k}_n$, components of (A 7) may be taken to give

$$\frac{\partial^2}{\partial t^2}\frac{\partial w_j}{\partial z} + gk_j^2 w_j = B_{mn}(t) \quad \text{at} \quad z = 0. \tag{A 8}$$

This equation represents the driving of the vertical velocity components of the jth wave by an effective surface force arising from the interactions of the mth and nth wave trains at second order.

For example, consider the interaction of two wave trains with vertical velocity components

$$w_1 = f_1(z, t)\cos(\mathbf{k}_1\,.\,\mathbf{x} - \sigma_1 t), \quad w_2 = f_2(z, t)\cos(\mathbf{k}_2\,.\,\mathbf{x} - \sigma_2 t) \tag{A 9}$$

to generate a third wave train of wave-number $\mathbf{k} = \mathbf{k}_1 + s\mathbf{k}_2$ and frequency $\sigma = \sigma_1 + s\sigma_2$, where s is $+1$ or -1. For simplicity we take f_1 and f_2 to be real.

It is found that the rate of change of the vertical component of vorticity at second order generated by these waves has an amplitude

$$\tfrac{1}{2}g\mu(\mathbf{k}_1\times\mathbf{k}_2)_z\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2}\right)f_1f_2 \quad \text{at level } z,$$

where

$$(\mathbf{k}_1\times\mathbf{k}_2)_z = k_{1x}k_{2y}-k_{2x}k_{1y}.$$

In this case (A 3) becomes

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_3}{\partial z^2}-k^2w_3\right)-g\mu k^2 w_3 = \left(\phi_1 f_1\frac{\partial f_2}{\partial z}+\phi_2 f_2\frac{\partial f_1}{\partial z}+\phi_3 f_1 f_2\right)ie^{-i\sigma t}, \qquad \text{(A 10)}$$

where

$$\phi_1 = -\tfrac{1}{2}g\mu\sigma\left\{-\frac{1}{\sigma_1^2k_2^2}[\mathbf{k}_1.\mathbf{k}_2(\mathbf{k}_1.\mathbf{k}_2+sk_1^2)-(\mathbf{k}_1\times\mathbf{k}_2)_z^2] +\frac{1}{\sigma_2^2}[k_2^2+s\mathbf{k}_1.\mathbf{k}_2]+\frac{sk^2}{\sigma}\left(\frac{1}{\sigma_2}-\frac{\mathbf{k}_1.\mathbf{k}_2}{k_2^2\sigma_1}\right)\right\},$$

$$\phi_2 = -\tfrac{1}{2}g\mu\sigma\left\{-\frac{1}{\sigma_2^2k_1^2}[\mathbf{k}_1.\mathbf{k}_2(\mathbf{k}_1.\mathbf{k}_2+sk_2^2)-(\mathbf{k}_1\times\mathbf{k}_2)_z^2] +\frac{1}{\sigma_1^2}[k_1^2+s\mathbf{k}_1.\mathbf{k}_2]+\frac{k^2}{\sigma}\left(\frac{1}{\sigma_2}-\frac{\mathbf{k}_1.\mathbf{k}_2}{k_1^2\sigma_2}\right)\right\},$$

and

$$\phi_3 = -\frac{g\dfrac{d\mu}{dz}\sigma^2}{2\sigma_1^2\sigma_2^2}\{\sigma_1k_2^2+s\sigma_2k_1^2+s\mathbf{k}_1.\mathbf{k}_2\sigma\}.$$

The upper boundary condition (A 8) becomes

$$\frac{\partial^2}{\partial t^2}\frac{\partial w_3}{\partial z}+gk^2w_3 = f_1f_2\phi_4ie^{-i\sigma t} \quad \text{at} \quad z=0, \qquad \text{(A 11)}$$

where

$$\phi_4 = \tfrac{1}{2}k^2\left\{\sigma\left(1+\frac{s\sigma_1}{\sigma_2}+\frac{s\sigma_2}{\sigma_1}\right)-\frac{g^2}{\sigma_1^2\sigma_2^2}\left[2s\sigma\mathbf{k}_1.\mathbf{k}_2+\left(\frac{\sigma_1-s\sigma_2}{\sigma_1\sigma_2}\right)(k_2^2\sigma_1^2-k_1^2\sigma_2^2)\right]\right\}.$$

Appendix 2

We wish to solve the equations of set III for w_l in the form $w_l = W_l'(z,t)\,e^{-i\sigma_l t}$, where $\sigma_l = \sigma_m \pm \sigma_n$ and $a_{mn}^{\pm}(z,t)$ is written $a_l(z,t)$ for convenience. There are two distinct classes of solution depending on whether or not $(\mathbf{k}_l, \sigma_l')$ is an eigenvalue of the equations of set I. We may expand $a_l(z,t)$ as a sum of eigenfunctions of set I for fixed $k^2/\sigma^2 = k_l^2/\sigma_l^2$. This is possible since the set of eigenfunctions, $W_n(z)$, say, with corresponding eigenvalues k_n, are complete;

$$a_l(z,t) = \sum_n \alpha_n(t)\,W_n(z).$$

The $\alpha_n(t)$ are given by

$$\alpha_n(t) = \int_{-H}^{0} a_l(z,t)\,W_n(z)\,dz \bigg/ \int_{-H}^{0} W_n^2(z)\,dz.$$

(An equivalent expansion in terms of eigenfunctions for fixed k^2 is also possible.)

Now suppose that the amplitudes of the wave trains present in the fluid, and therefore the maximum values (the amplitudes) of their vertical velocity components, change only slowly as a result of the interactions. This leads to an approximation equivalent to that made by Benney in the analysis of surface interactions. We suppose that the change in amplitude of the vertical velocity components during one oscillation of the wave is much less than the mean amplitude of the vertical velocity component itself during the oscillation.† If this approximation is made, and it may be seen *a posteriori* that it is a valid approximation here, the solutions of set III are as follows.

(1) *If* (k_l, σ_l) *is not an eigenvalue of set I; boundary condition* (b_1)

$$w_l = \sum_n \frac{\alpha_n W_n}{\sigma_l^2(k_l^2-k_n^2)} e^{-i\sigma_l t}.$$

(2) *If* (k_l, σ_l) *is not an eigenvalue of set I; boundary condition* (b_2)

$$w_l = \left(\sum_n \frac{\alpha_n W_n}{\sigma_l^2(k_l^2-k_n^2)} + U(z,t)\right) e^{-i\sigma_l t},$$

where $U(z,t)$ satisfies

$$\frac{\partial^2}{\partial t^2} w - k_l^2 w + g\mu \frac{k_l^2}{\sigma_l^2} w = 0,$$

with boundary conditions

$$w = 0 \quad \text{at} \quad z = -H$$

and

$$\frac{\partial w}{\partial z} - \frac{gk_l^2}{\sigma_l^2} w = -\frac{b(t)}{\sigma_l^2} \quad \text{at} \quad z = 0.$$

(3) *If* (k_l, σ_l) *is an eigenvalue of set I with corresponding eigenfunction* $W_N(z)$ *and boundary condition* (b_1)

$$w_l = (\alpha(t) W_N + \sum_{n \neq N} a_n(t) W_n) e^{-i\sigma_l t},$$

where, if

$$\frac{d^2 W_N}{dz^2} - k^2 W_N = \sum_n \beta_n W_n(z),$$

$$\frac{d\alpha}{dt} = \dot{\alpha} = \frac{i\alpha_N}{2\beta_N \sigma_l}$$

and

$$a_n \sigma_l^2(k_l^2 - k_n^2) - 2i\dot{\alpha}\beta_n = \alpha_n \quad \text{if} \quad n \neq N.$$

Here we note that $\alpha(t)$ is not determined uniquely. Only $\dot{\alpha}(t)$ is determined by the boundary conditions. $\alpha(t)$ is in fact determined fully by the initial conditions and the subsequent value of $\dot{\alpha}(t)$ found above. Moreover, if $\alpha_N/2\beta_N$ is real, then $\dot{\alpha}$ is pure imaginary and the effect of the interaction is simply to change the phase of the wave corresponding to the eigenfunction W_N; only if $\alpha_N/2\beta_N$ is pure imaginary does a pure growth in amplitude occur. $\dot{\alpha} = 0$ if $\alpha_N = 0$, that is if the coefficient of the eigenfunction W_N corresponding to the eigenvalue (k, σ) in the expansion of the term $a(z,t)$ as a series of the eigenfunctions is zero.

† Some adjustment of this statement is obviously required when one of the velocity amplitudes is initially zero or becomes zero. This is the situation when wave generation occurs, that is when initially no wave exists with the frequency and wavelength excited by two interacting wave trains. If it is found that the solutions are not affected by such a vanishing of the amplitude of the vertical component of a wave velocity, since it is the ratio of the wave period to the time scale of the amplitude changes which must be small. The solutions are valid even when w_l is initially zero, provided that the terms $\alpha_n(t)$ are slowly varying functions of time.

(4) *If* (k_l, σ_l) *is an eigenvalue of set I with corresponding eigenfunction* $W_N(z)$ *and boundary condition* (b_2)

$$w_l = \left(\alpha(t)\, W_N + \beta V + \gamma W^* + \sum_{n \neq N} a_n(t)\, W_n\right) e^{-i\sigma t},$$

where $V(z)$ is a particular integral of

$$\frac{d^2 w}{dz^2} - k_l^2 w + g\mu \frac{k_l^2}{\sigma_l^2} w = W_N,$$

which is linearly independent of W_N, and W^* is a solution of

$$\frac{d^2 w}{\partial z^2} - k_l^2 w + g\mu \frac{k_l^2}{\sigma_l^2} w = 0,$$

which is linearly independent of W_N. (This second-order differential equation for w has, in general, two independent solutions for given k_l and σ_l. One satisfies both boundary conditions of set I. The other does not.) If β_n is defined as in (3) above, then α is given by

$$\dot{\alpha}(t) = \frac{i\left\{b(t) + \alpha_N\left[\frac{gk_l^2}{\sigma_l^2} V(0) - V'(0) - \frac{V(-H)}{W^*(-H)}\left(\frac{gk_l^2}{\sigma_l^2} W^*(0) - \frac{dW^*(0)}{dz}\right)\right]\right\}}{2\sigma_l\left\{\frac{dW_N(0)}{dz} + \beta_N\left[\frac{gk_l^2}{\sigma_l^2} V(0) - V'(0) - \frac{V(-H)}{W^*(-H)}\left(\frac{gk_l^2}{\sigma_l^2} W^*(0) - \frac{dW^*(0)}{dz}\right)\right]\right\}},$$

where $V(0)$ is the value of $V(z)$ at $z = 0$, etc.,

$$\beta = -\frac{\alpha N}{\sigma_l^2} - \frac{2i\dot{\alpha}}{\sigma_l}\beta_N,$$

$$\gamma = -\beta \frac{V(-H)}{W^*(-H)},$$

and $$a_n \sigma_l^2 (k_l^2 - k_n^2) - 2i\sigma_l \beta_n \dot{\alpha} = \alpha_n \quad (n \neq N).$$

As in solution (3), the coefficient $\alpha(t)$ is not fully determined.

Appendix 3

This is an appropriate place to correct a slight inaccuracy which has been found in Ball's (1964) paper. In the discussion of interactions between a pair of surface waves and an interfacial wave in a two-fluid system, interactions between the waves when all are moving in the same direction are excluded. To exclude this possibility in general is clearly incorrect, for if the depths of the two fluids are both large compared with the wavelength present and the fluid densities are ρ_1 and $\rho_2\,(\rho_1 < \rho_2)$, both the dispersion relations (which are $\sigma_i^2 = gk_i, i = 1, 2$, for surface waves and $\sigma_3^2 = gk_3(\rho_2 - \rho_1)/(\rho_1 + \rho_2)$ for interfacial waves), and the resonance conditions for parallel waves ($\sigma_1 - \sigma_2 = \sigma_3, k_1 - k_2 = k_3$), are satisfied by

$$\sigma_1 = \frac{\rho_2}{\rho_1}\sigma_2, \quad k_1 = \frac{\rho_2^2}{\rho_1^2}k_2, \quad \sigma_3 = \left(\frac{\rho_2}{\rho_1} - 1\right)\sigma_2, \quad k_3 = \left(\frac{\rho_2^2}{\rho_1^2} - 1\right)k_2,$$

for any σ_2, k_2 which satisfy the surface dispersion relation. For interactions to occur between a given surface wave and another surface wave and an interfacial wave, all travelling in the same direction, it is necessary that the group velocity of the given surface wave ($d\sigma/dk$, the slope of the curve OE_1, in figure 1 of Ball's paper) shall be less than the limit of the group velocity of interfacial waves as their wave-number tends to zero, as may be seen from an examination of Ball's interaction diagram. It seems unlikely, however, that this case is of much practical importance, either because the circumstances in which the conditions are satisfied do not occur naturally, or because the interaction coefficients which determine the rate of energy transfer between the waves are so small.

REFERENCES

BALL, F. K. 1964 *J. Fluid Mech.* **19**, 465.

BENNEY, D. J. 1962 *J. Fluid Mech.* **14**, 577.

BRETHERTON, F. P. 1964 *J. Fluid Mech.* **20**, 401.

CHERKESOV, L. F. 1962 *Prik. Mat. Mek.* **26**, 559.

COX, C. & SANDSTROM, H. 1962 *J. Oceanographic Soc., Japan*, **20**, 499.

DRAZIN, P. 1958 *J. Fluid Mech.* **4**, 214.

FJELDSTAD, J. E. 1933 *Geofysiske Publikasjoner*, **10**, 1.

FRASSETTO, R., BACKUS, R. H. & HAYS, E. 1962 *Deep Sea Research*, **9**, 69.

GROEN, P. 1948 *Physica*, **14**, 294.

GROEN, P. & HEYNA, B. 1958 *Physica*, **24**, 383.

HASSELMANN, K. 1962 *J. Fluid Mech.* **12**, 481.

HASSELMANN, K. 1963 *J. Fluid Mech.* **15**, 273.

HASSELMANN, K. 1963 *J. Fluid Mech.* **15**, 385.

HOWARD, L. N. 1961 *J. Fluid Mech.* **10**, 509.

LAFOND, E. C. 1959 U.S. Navy Electronics Lab., San Diego, California. *Research Rep.* no. 937.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LONG, R. R. 1964 Johns Hopkins Univ., Dept. Mech. *Tech. Rep.* no. 17 (O.N.R. Series), (and *Tellus*, **17**, 46).

LONGUET-HIGGINS, M. S. 1962 *J. Fluid Mech.* **12**, 321.

LONGUET-HIGGINS, M. S. & PHILLIPS, O. M. 1962 *J. Fluid Mech.* **12**, 333.

MCGOLDRICK, L. F. 1965 *J. Fluid Mech.* **21**, 305.

METCALF, W. G., VOORHIS, A. D. & STALCUP, M. C. 1962 *J. Geophys. Res.* **67**, 2499.

MILES, J. W. 1961 *J. Fluid Mech.* **10**, 496.

PHILLIPS, O. M. 1960 *J. Fluid Mech.* **9**, 193.

RATTRAY, M. 1960 *Tellus*, **12**, 54.

SANDSTRÖM, J. 1908 *Ann. Hydr. Mar. Met.* **36**, 6.

SUMMERS, H. J. & EMERY, K. O. 1963 *J. Geophys. Res.* **68**, 827.

TAYLOR, G. I. 1931 *Proc. Roy. Soc.* A, **132**, 499.

YANOWITCH, M. 1960 *IMM-NYU*, p. 268.

YANOWITCH, M. 1962 *Comm. Pure & Appl. Maths.* **15**, 45.

YIH, C. S. 1960 *J. Fluid Mech.* **8**, 481.